Origami

Introduction

Origami is the ancient art of paper folding and whilst this technique is often associated with Japanese culture, its exact origins are unclear. In fact, paper folding has a history that spans a wide range of different cultures from across the globe. However, the true beauty of origami lies not only in the fact that it can be used to create magical structures and paper animals, but it also has applications in mathematics. The rules of paper folding, for example, are encoded in several mathematical axioms known as the Huzita–Hatori axioms. Several geometric construction problems, such as trisecting an angle or doubling a cube, can also be solved using only a few paper folds.

The mathematics of origami has also been extended to real-world applications. The Miura fold, for example, is a highly effective way of folding a piece of paper into a much smaller area and is implemented in space missions to deploy solar panels. While a high school student, Britney Gallivan derived a formula for the maximum number of times a piece of paper can be folded in half and proved the answer to be 12.

Aim of Workshop

The aim of this workshop is to demonstrate mathematical proofs using origami, which the students can physically construct and observe. In particular, students will have the opportunity to prove Pythagoras' theorem using a single sheet of origami paper. The limitations of traditional constructions using a straightedge and compass will also be explored.

Learning Outcomes

By the end of this workshop, students will be able to:

- Construct squares and triangles to prove Pythagoras' theorem
- Justify why $\sqrt[3]{2}$ is not constructible using a straightedge and compass
- Construct $\sqrt[3]{2}$ using origami
- Recognise that several proofs to one problem may exist

Materials and Resources

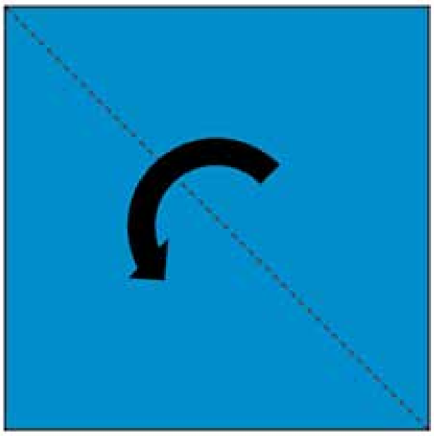
Origami paper, activity sheets

Origami: Workshop Outline

Suggested Time (Total mins)	Activity	Description
5 mins (00:05)	Introduction to the art of paper folding	· Introduce the mathematics of origami (see **Workshop Introduction)**. · Mention that the aim of the workshop is to prove a prominent theory in mathematics using origami
20 mins (00:25)	Activity 1 Proof by origami	· Demonstrate the folding technique by following the steps alongside the students (see **Appendix – Note 1**) · **Activity Sheet 1:** Using their folded origami paper, students attempt to prove the theorem of Pythagoras (see **Appendix – Note 2**)
5 mins (00:30)	Doubling a cube	· Introduce the 'doubling a cube' problem, also known as the Delian problem (see **Appendix – Note 3**)
15 mins (00:45)	Activity 2 Calculating the volumes of cubes	· **Activity Sheet 2:** Students calculate the volumes of various cubes (see **Appendix – Note 4**) · Note: The final questions involve cubed roots and may need to be revised following a discussion of roots
10 mins (00:55)	$\sqrt{2}$ and $\sqrt[3]{2}$	· Ask the students to discuss whether $\sqrt{2}$ is constructible with a straightedge and compass (It is – we can use Pythagoras' theorem!) · Explain that $\sqrt[3]{2}$ cannot be constructed with a straightedge and compass – link back to Activity Sheet 2 (see **Appendix – Note 3**)
5 mins (01:00)	Activity 3 Constructing $\sqrt[3]{2}$ with origami	· Use origami to demonstrate the "construction" of $\sqrt[3]{2}$ by a marked straightedge · Encourage students to measure lengths a and b, and to calculate the ratio.

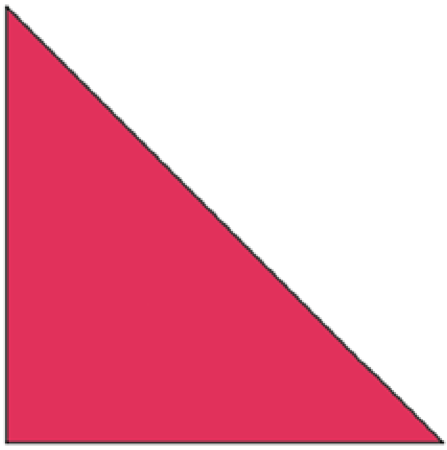
Note 1: Origami Paper Folding Instructions

1



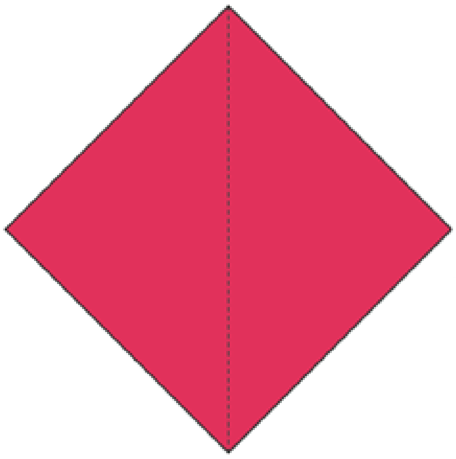
Fold your paper in half along the dotted line as shown.

2



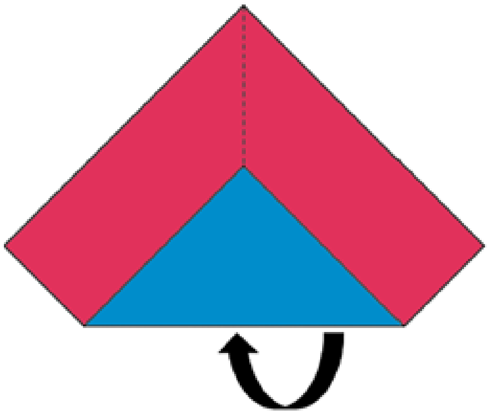
This will be the result.

3



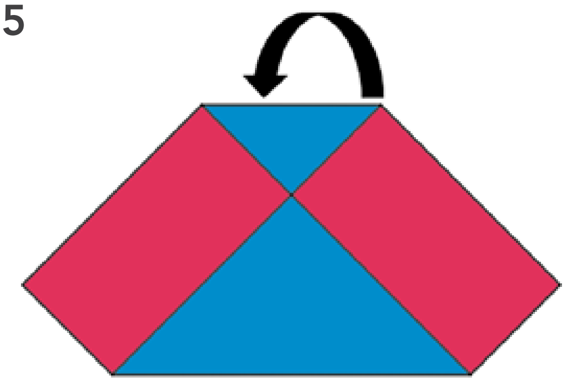
Open the paper and position it like this.

4



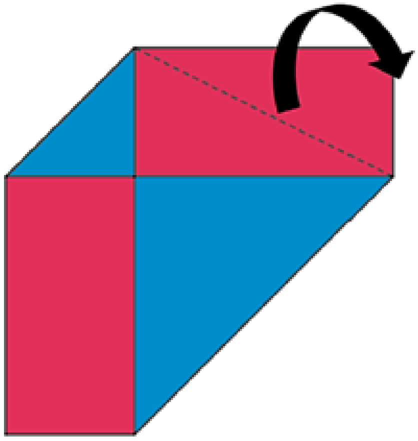
Fold up from the bottom. The triangle you make can be any size that is less than half your original paper.

5



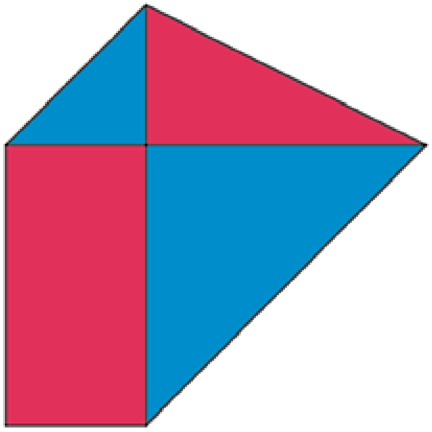
Fold down from the top. The new triangle needs to meet the triangle from the previous step.

6



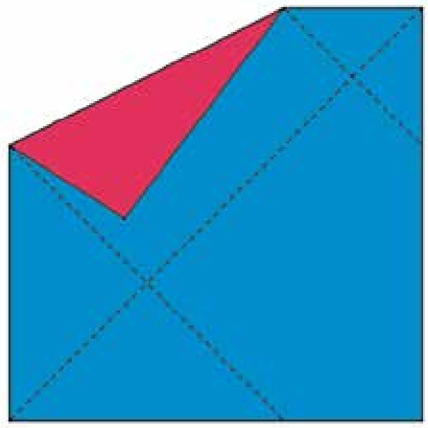
Rotate your paper. Fold along the dotted line to put a triangle behind your paper.

7



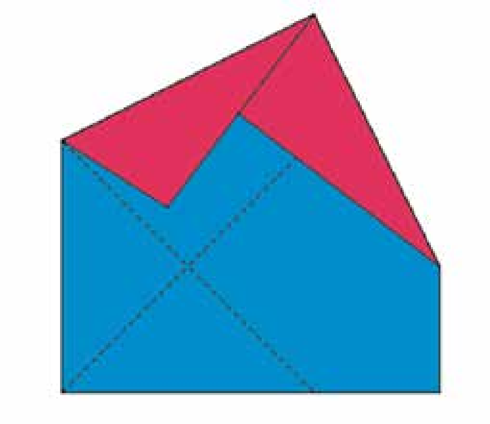
This is what you should have now after folding back the triangle.

8



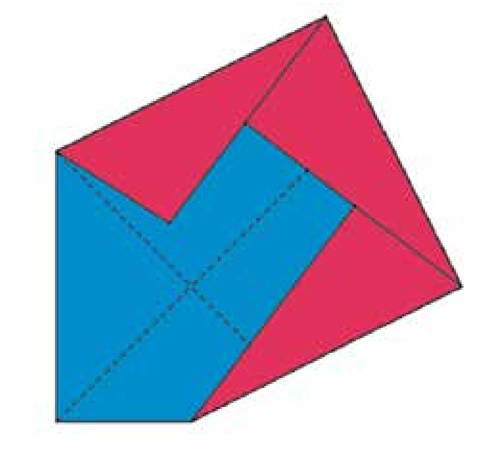
Flip over your paper. Unfold everything except the triangle you just made.

9



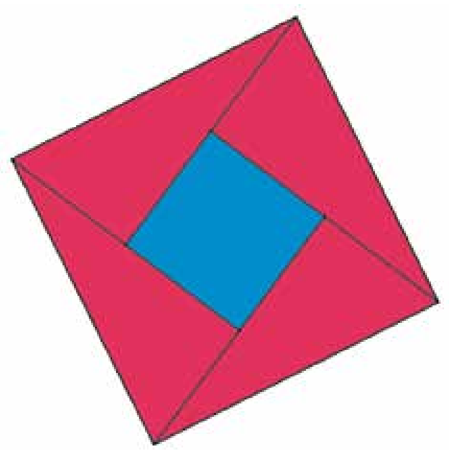
Fold down the top right hand corner to make a triangle. Ensure that the side of the new triangle matches up with the side of the other triangle.

10



Repeat step 9 with the bottom right corner to make another triangle.

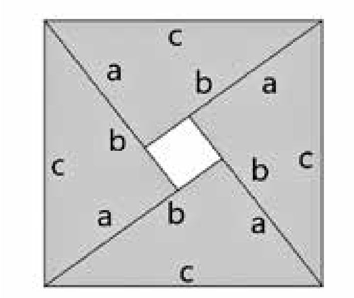
11



Fold in the last corner to make a fourth triangle. Again, make sure that the sides of the triangles meet each other. This is your final product!

Note 2: Solutions for Activity 1

Q1. Fill in the blanks:



(i) A right-angled triangle is a triangle that has one angle measuring 90 degrees.

(ii) The side of the triangle opposite the right angle is the longest side and is called the hypotenuse.

Q2. What is the area of the bigger square (the entire square piece of paper you ended up with after folding)?

The area of the bigger square is c^2 since each side of the square is length c

Q3. What is the area of the smaller square?

$(b - a)^2$

Q4.

(i) What is the area of one of the four identical triangles?

1/2 (ab) (Recall; the area of a triangle can be calculated by multiplying the base by the perpendicular height and dividing by two)

(ii) What is the area of the four triangles together?

2ab

Q5. In words, what is the relationship between the area of the bigger square, the area of the smaller square and the area of the four triangles?

Area of bigger square = Area of smaller square + Area of four triangles

Q6. Use this relationship and your other answers to derive an important theorem in mathematics!

$c^2 = (b - a)^2 + 2ab = b^2 - 2ab + a^2 + 2ab = a^2 + b^2$

(Pythagoras' theorem)

Note 3: Doubling a Cube

'Doubling a cube' (or the Delian Problem) is an ancient geometric problem which involves the construction of the edge of a cube whose volume is double that of a given cube.

This problem owes its name to a Greek legend concerning the inhabitants of Delos who were suffering from a terrible plague that was ravaging their island. The inhabitants believed that Apollo, the god of healing, had purposely sent the plague to kill them and so they sought guidance from the oracle (priest) at Delos on how to appease the gods. The oracle explained that Apollo was furious because the altar in the temple of Delos was too small and therefore instructed them to double it in size. The people of Delos immediately rushed to the temple to construct a new alter that was twice as wide, twice as long and twice as tall as the previous one in the hope of saving their island. However, Apollo was still not pleased as the new altar was now eight times the size of the original, rather than twice the size. As a consequence, the plague continued to spread throughout the island of Delos, claiming the lives of its inhabitants.

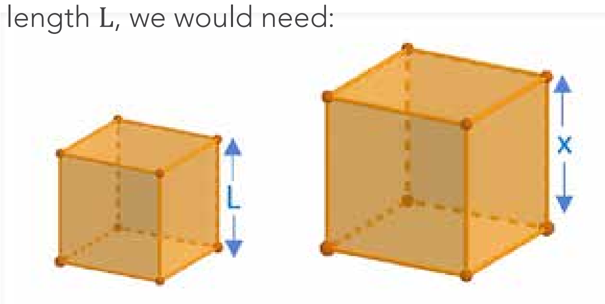
This legend demonstrates that doubling the dimensions of a cube will not double the volume.

In order to double the volume of a cube, with side length L, we would need:

- (Length of larger cube)$^3 = 2(L^3)$
- Let the length of the larger cube be x.

We, therefore, have that $x^3 = 2(L^3)$

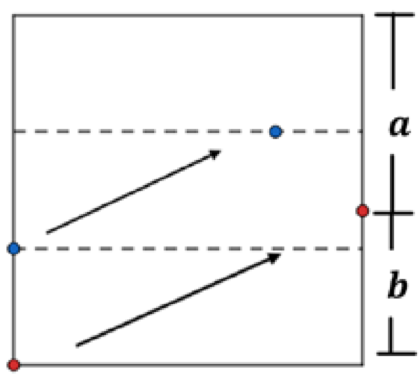
- Solving for x, we see that $x = \sqrt[3]{2}\ L$.



In other words, the length of the new cube would need to be $\sqrt[3]{2}$ times the length of the original cube. However, $\sqrt[3]{2}$ is an irrational number ($\approx 1.259921\ldots$) and we, therefore, cannot accurately construct a cube that is double the volume of another using a straightedge and compass.

However, it is possible to construct this ratio using origami:

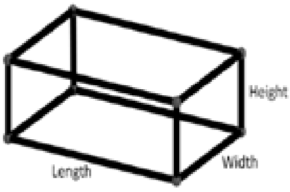
1. Fold a square piece of paper into thirds as shown below.
2. Fold the paper so that the bottom third (blue dot) touches the two-thirds line at the same time as the bottom left-hand corner (red dot) touches the right edge of the paper
3. The ratio between a and b is $\sqrt[3]{2}$. Therefore, a cube with a side length of "a", will have twice the volume of a cube of side length "b". We can see this by rearranging $a/b = \sqrt[3]{2}$ to get $a = \sqrt[3]{2}\ (b)$



$$\frac{a}{b} = \sqrt[3]{2}$$

Note 4: Solutions for Activity 2

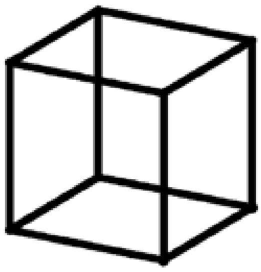
Q1. Label the edges of the cuboid with the following terms: length, width, height



Q2. What is the volume formula for a cuboid?

$\text{Volume of cuboid} = \text{Length} \times \text{Width} \times \text{Height}$

Q3. Label the edges of the cube



Q4. What is the volume formula for a cube?

$\text{Volume of cube} = \text{Length} \times \text{Length} \times \text{Length} = \text{Length}^3$

Q5. What is the volume of the following cube?

$(4\text{ cm})^3 = 64\text{ cm}^3$

Q6. What would the volume of a cube with double (or twice) the volume of the previous cube be?

$(64\text{cm}^3) \times 2 = 128\text{ cm}^3$

Q7. What would the volume of a cube with double the length of the cube in Q5 be?

$(4\text{ cm} \times 2)^3 = (8\text{cm})^3 = 512\text{ cm}^3$

Q8. What do you notice about the answers to Q6 and Q7?

The answers are different; we cannot double the volume of a cube just by doubling its length.

Q9. What would the volume of a cube with double the volume of the following cube be (in terms of L)?

$\text{Volume} = 2L^3$

Q10. We have a cube with a side of length y cm. We want our cube to have the same volume as the cube in the answer to Q6. What do we need the value of y to be?

$y = \sqrt[3]{128} \approx 5.04$

Q11. We have a cube with a side of length x cm. Now we want our cube to have the same volume as the cube in the answer to Q9. What do we need the value of x to be?

$x = \sqrt[3]{(2L^3)} = \sqrt[3]{2}\ (L)$

Sources and Additional Resources

https://www.tor.com/2017/06/29/the-magic-and-mathematics-of-paper-folding/ (Mathematics of origami)

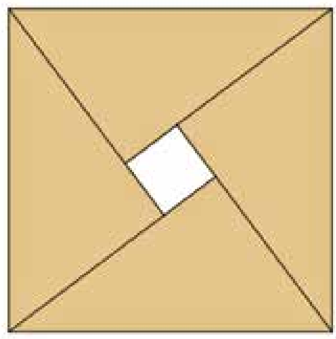
https://youtu.be/R4IMaeZmgLA (Proof of Pythagoras using origami)

https://www.youtube.com/watch?v=4Ncc5A2xT78 (Delian problem)

http://www.cutoutfoldup.com/409-double-a-cube.php (Constructing $\sqrt[3]{2}$ using origami)

Origami: Activity 1

Having completed the folds, you should end up with something like this. Don't worry if your inner square is bigger or smaller than someone else's!



Q1. Fill in the blanks:

(i) A right-angled triangle is a triangle that has one angle measuring _______________degrees.

(ii) The side of the triangle opposite the right angle is the longest side and is called the _________________

Instructions:

- Label this longest side in each of the four triangles you have folded **c**. These will also be the four sides of the large square.
- Label the other two sides in the four triangles **a** and **b**. Call the smallest side **a** and the other side **b**.

Q2. What is the area of the bigger square (the entire square piece of paper you ended up with after folding) in terms of the variables above?

Q3. What is the area of the smaller square in terms of the variables?

Q4.

(i) What is the area of one of the four identical triangles in terms of the variables?

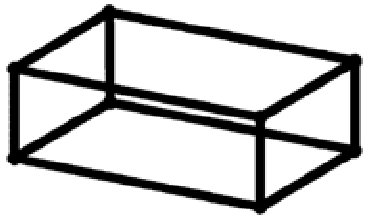
(ii) What is the area of the four triangles together in terms of the variables?

Q5. In words, what is the relationship between the area of the bigger square, the area of the smaller square and the area of the four triangles?

Q6. Use this relationship and your other answers to derive an important theorem in maths!

Origami: Activity 2

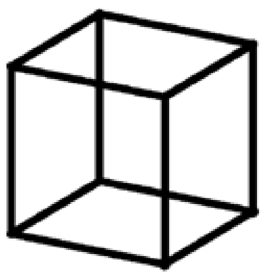
Q1. Label the edges of the cuboid with the following terms: Length, Width, Height



Q2. What is the volume formula for a cuboid?

Volume of cuboid =

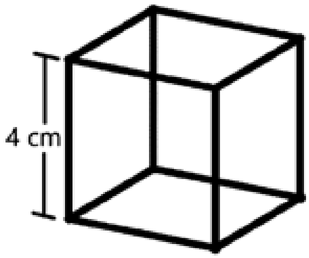
Q3. Label the edges of the cube.



Q4. What is the volume formula for a cube?

Volume of cube =

Q5. What is the volume of the following cube?



Q6. What would the volume of a cube with double (or twice) the volume of the previous cube be?

Q7. What would the volume of a cube with double the length of the cube in Q5 be?

Q8. What do you notice about the answers to Q6 and Q7? Are they the same?

Q9. What would the volume of a cube with double the volume of the following cube be in terms of L?

Q10. We have a cube with a side of length y cm. We want our cube to have the same volume as the cube in the answer to Q6. What do we need the value of y to be?

Q11. We have a cube with a side of length x cm. Now we want our cube to have the same volume as the cube in the answer to Q9. What do we need the value of x to be?